then $\bar{e}$ is given in Ref. 1 by 0.8981. Our investigation shows that the least conservative value of $\bar{e}$ is

$$\bar{e} = 1 \tag{7}$$

That is, the method proposed in Ref. 1 results in a conservative estimate of the maximum allowed perturbation.

It is worth noting here that the stability-margin problem as stated above can be formulated as that of finding a zero of a nonlinear (also, nondifferentiable) function in one parameter. Namely, let

$$f(e) = \max[R\lambda(A + eE)] \tag{8}$$

where $\lambda(\ )$ corresponds to the set of eigenvalues of the given matrix and R stands for taking the real part of a complex number. Then, the minimum value $|\bar{e}|$ such that $f(\bar{e}) = 0$ is actually the stability margin that we are seeking. We have successfully checked the results in the above two counterexamples using a simple bisection algorithm to find the zeros of the function f in Eq. (8).

Acknowledgment

This work was supported by Systems Control Technology, Inc., internal research and development funds under Project 8553-890.

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Reply by Author to Nasser M. Khraishi

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OUR paper[1] deals with a special type of perturbations that move in a certain fixed direction $\bar{E}$ by the same relative scaling e ($0 \le e \le \bar{e}$). As pointed out in the paper, the main reason for choosing such a structure for the perturbation matrix is the need to incorporate the directional information in the perturbation matrix, which was not the case with the perturbation characterization proposed in Refs. 2-4. In this way, we considerably improved the perturbation bounds.[2-4] However, it should be pointed out that our approach can also be applied to the more general cases of perturbation characterizations proposed in Refs. 2-4. Furthermore, although it may seem that our assumption is quite restrictive, there are at least two classes of systems widely used in practice, where the perturbation directions are well-defined:

1) Singularly perturbed systems,[5-7] where the directional perturbations in the open-loop matrix A and in the control actuating matrix B, are defined by

$$\Delta A = \frac{1}{\lambda}\begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix}, \qquad \Delta B = \frac{1}{\lambda}\begin{bmatrix} 0 \\ B_2 \end{bmatrix}$$

2) Weakly coupled systems[6]

$$\Delta A = \epsilon\begin{bmatrix} 0 & A_{12} \\ A_{21} & 0 \end{bmatrix}, \qquad \Delta B = \epsilon\begin{bmatrix} 0 & B_{12} \\ B_{21} & 0 \end{bmatrix}$$

As is well known, the entries of these matrices are known, and are not necessarily positive numbers, and they cannot be properly characterized by the approaches proposed in Refs. 2–4, where e_{ij} are assumed to be positive numbers.

For a more general case, when every element of $\bar{E}$ may undergo changes in an independent and possibly random fashion, we proposed a new approach on stability robustness characterization, based on the concept of interval matrices.[8] However, although a wide choice of measures of the "size" of the perturbations has been suggested in the area of robustness analysis for multivariable systems, a basic need still remains for more refined tests and measures of robustness.

As in the case of most of the results published in the literature, our stability robustness criterion only provides *sufficient conditions*, and assumes that $e \ge 0$. However, it still represents an improvement over the results published in Refs. 2-4. We have not claimed that our algorithm leads to exact stability margins.

Therefore, the so-called counterexamples only pointed out that we have proposed only sufficient conditions, restricted to the case when $e \ge 0$, and that further research is needed to overcome the conservatism of the existing stability robustness criteria for multivariable control systems.

The case when e is not restricted to being a positive number can easily be included by using the following theorem:

Theorem 1. The perturbed system

$$\dot{x}(t) = (A + e\bar{E})x(t) \tag{1}$$

where A is asymptotically stable matrix and $\bar{E}$ is a perturbation matrix, is asymptotically stable if the following inequalities are satisfied:

$$\lambda_{\min}^{-1}(Y) = e_{\min} < e < e_{\max} = \lambda_{\max}^{-1}(Y) \tag{2}$$

where $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ are the minimum and maximum eigenvalues of $(\cdot)$, respectively, and

$$Y = \tfrac{1}{2}(\bar{E}^T P + P\bar{E}) \tag{3}$$

and the matrix P is the positive definite solution of

$$A^T P + PA + 2I = 0 \tag{4}$$

Proof. The proof proceeds by using the argument of Lyapunov theory. Consider the positive definite function $V(x)$ for the perturbed system

$$V(x) = x^T P x \tag{5}$$

Since P is the positive definite matrix, it remains to examine $\dot{V}(x)$. Taking the time derivative of $V(x)$ along the solution of Eq. (1), it follows that

$$\dot{V}(x) = x^T[2I - e(\bar{E}^T P + P\bar{E})]x \tag{6}$$

making the simplification by using the Lyapunov function, Eq. (4).

Asymptotic stability follows if $\dot{V}(x)$ is negative definite, which follows if

$$2I - e(\bar{E}^T P + P\bar{E}) > 0 \tag{7}$$

To prove the conditions given by Eq. (2), recall the following lemma.

Received March 25, 1990; accepted for publication May 2, 1990.
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Lemma 1.[9] If R and S are symmetric matrices and R is positive definite, there exists a nonsingular matrix W such that

$$W^T(R + S)W = I + G \tag{8}$$

where matrix G is a diagonal matrix whose elements are eigenvalues of $R^{-1} S$.

Therefore, using the results of Lemma 1, it can be easily concluded that the perturbed system, Eq. (1), will remain asymptotically stable if the following inequality is satisfied:

$$1 - e\lambda_i[½(\bar{E}^TP + P\bar{E})] > 0, \qquad i = 1,2, \ldots, n \tag{9}$$

i.e.,

$$1 - e\lambda_i(Y) > 0, \qquad i = 1,2, \ldots, n \tag{10}$$

where the matrix Y is defined by Eq. (3).

Now, under the assumption that $\lambda_{\max}(Y) > 0$, which is a usual case, it follows that

$$e_{\max} = \lambda_{\max}^{-1}(Y)$$

In a similar way, if $\lambda_{\min}(Y) < 0$, then

$$e_{\min} = \lambda_{\min}^{-1}(Y)$$

i.e., the perturbed system, Eq. (1), remains asymptotically stable if

$$e \epsilon (e_{\min}, e_{\max})$$

If we apply Theorem 1 to perturbation matrix

$$\bar{E} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

we get the following bounds

$$-0.3098431 < e < 0.8098431$$

The iterative algorithm proposed in Ref. 1 can easily be extended to incorporate the results of Theorem 1. In this case we get

$$-0.999999 < e < \infty$$

Singular results could be achieved for the other perturbation matrices given in Ref. 1. For example,

$$\bar{E} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \bar{e} = 0.9, \qquad -\infty < e < 0.9999172$$

$$\bar{E} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \bar{e} = 2.9981, \qquad -0.9999991 < e < 2.999936$$

$$\bar{E} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad \bar{e} = 0.8981, \qquad -0.99832 < e < 0.99817$$

Therefore, if one uses the algorithm proposed in Ref. 1, but based on Theorem 1, further improvement of the stability robustness bounds could be achieved.

When talking about the convergence of the proposed algorithm, one should notice that it provides a sequence S_i of "perturbed" systems and a sequence e_i of scalars. Furthermore, for each i, the eigenvalues of the corresponding system (S_i)

$$A = A + \sum_{p=0}^{i-1} e_p \bar{E}$$

have negative real parts. If, for any i, there is an eigenvalue with zero real part, we will not be able to apply Theorem 1 and we shall have $\bar{e} = e_i$. If $p \to \infty$, then the corresponding Lyapunov equation becomes progressively more ill-conditioned as $p \to \infty$ and the procedure will have to stop for some finite value of e_i.

Notice that $e_p > 0$, $p = 0,1, \ldots, i-1$. Therefore,

$$e_{i-1} = \sum_{p=0}^{i-2} < e_p < e_i = \sum_{p=0}^{i-1} e_p$$

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